COSET GEOMETRIES OF SOME GENERALIZED SEMI-DIRECT PRODUCTS OF GROUPS

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ABSTRACT

A generalization of the standard semi-direct product of groups is given. The following special case is exploited in the construction of partial 4-gons. Let G be the set of 4-tuples of elements of the finite field F. For all i, j with $1 \leq i, j \leq 2$, let L_{ij} and R_{ij} be linear transformations of F over its prime subfield. Then define a product on G as follows:

$$(a_1, b_1, c_1, d_1)\cdot(a_2, b_2, c_2, d_2) = (a_1+a_2, b_1+b_2,$$

$$a_1^{L_{11}} b_2^{R_{11}} + a_2^{L_{12}} b_1^{R_{12}} + c_1+c_2, a_1^{L_{21}} b_2^{R_{21}} + a_2^{L_{22}} b_1^{R_{22}} + d_1+d_2).$$

With this product G is a group. Let A and B be the subgroups of G consisting of elements of the form $(a, 0, 0, 0)$, $a \in F$, and $(0, b, 0, 0)$, $b \in F$, respectively. Then necessary and sufficient conditions on L_{ij} and R_{ij} are found for the coset geometry $\pi(G, A, B)$ to be a partial generalized 4-gon.

INTRODUCTION

In a recent article, Sah (1968) gave the following simple construction of a nonabelian group. Let R be a ring, and put $G = \{(a, b, c) \mid a, b, c \in R\}$. Define a product on G by $(a, b, c)\cdot(x, y, z) = (a+x, b+y, ay+c+z)$. It is then an easy exercise to show that G with this product is a group and will be nonabelian unless the ring product is trivial. An interesting special case is obtained by letting R be the integers modulo p, p prime. Then G is the nonabelian group of order p^3 in which each non-identity element has order p.

Actually the associative law for ring multiplication is never invoked while verifying that G is a group, and the above construction is easily generalized. Indeed, it has been in the theory of factor sets as presented in MacLane (1963). The generalization to be considered here amounts to choosing "nice" factor sets. And the reason for considering these particular groups is that certain examples have coset geometries which are partial generalized 4-gons as introduced in the last section of this paper. In some instances these partial 4-gons may be completed to 4-gons. The real goal is to construct generalized 4-gons, and some progress related to this attempt is reported in Payne (1970). Before discussing the 4-gons, it is necessary to recall some facts about pairings of groups.

PAIRINGS OF GROUPS

Let A, B, and C be groups whose operations are written multiplicatively.

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A *pairing* of A and B into C is a mapping $(\ ,\ ) : A\times B \to Z(C)$ (here $Z(C)$ denotes the center of C) such that

i) $(a_1a_2, b) = (a_1, b)\cdot(a_2, b)$ for all $a_1, a_2 \in A$, $b \in B$

ii) $(a, b_1b_2) = (a, b_1)\cdot(a, b_2)$ for all $a \in A$, $b_1, b_2 \in B$.

A number of results suggest themselves immediately. If $(\ ,\ )$ is a pairing of A and B into C, the map $(a, -)$ from B into C defined by $(a, -)(b) = (a, b) \in Z(C)$ is an element of the abelian group Hom(B, Z(C)) with pointwise product. Similarly, the map $a \to (a, -)$ is an element of the abelian group Hom(A, Hom(B, Z(C))). Conversely, let $a \to \hat{a}$ be a homomorphism of A into Hom(B, Z(C)). Then a pairing of A and B into C is obtained by putting $(a, b) = b^{\hat{a}}$, for

$$(a_1a_2, b) = b^{\widehat{a_1a_2}} = b^{\hat{a}_1}\, b^{\hat{a}_2} = (a_1, b)\cdot(a_2, b).$$

And clearly $(a, b_1b_2) = (b_1, b_2)^{\hat{a}} = b_1^{\hat{a}} b_2^{\hat{a}} = (a, b_1)\cdot(a, b_2)$. This essentially shows that the correspondence $(\ ,\ ) \longleftrightarrow {}^\wedge$ determined by $(a, -) \longleftrightarrow \hat{a}$ is an isomorphism. Thus the set of pairings of A and B into Z(C) with the product $(\ ,\ )_1\cdot(\ ,\ )_2 = (\ ,\ )_3$ defined by $(a, b)_3 = (a, b)_1\cdot(a, b)_2$ is a group isomorphic to Hom(A, Hom(B, Z(C))).

If D and E are groups with E abelian, then Hom(D, E) is easily shown to be isomorphic to Hom(D$'$, E), where D$'$ is the quotient group D modulo its commutator subgroup. Since for abelian groups Hom distributes over direct sums, it is not too difficult to determine the invariants of Hom(A, Hom(B, Z(C))) $\cong$ Hom(A$'$, Hom(B$'$, Z(C))) in case A, B, and C are finite groups. In particular there will be a nontrivial pairing of A and B into C (in the finite case) if and only if $\gcd(|A'|, |B'|, |Z(C)|) > 1$.

For the present it is important only that the reader have in mind several examples of pairings. As one natural example, let A be a ring and B a left A-module. Then the module multiplication $(a, b) = ab$ is a pairing of (the additive group of) A and B into B, and nowhere is the associativity of multiplication of A used.

Suppose that $b \to \hat{b}$ is a representation of B as a group $\hat{B}$ of automorphisms of A operating on the left, i.e. $\widehat{b_1b_2}\cdot a = \hat{b}_1\cdot(\hat{b}_2\cdot a)$. A pairing of A and B into a group C is said to be *compatible* with the representation $b \to \hat{b}$, provided $(\hat{b}\cdot a, b') = (a, b')$ for all $a \in A$, $b, b' \in B$. With this definition, the general construction may be given.

THE GROUPS $G(A, B, \{C_\alpha\})$

Throughout this section let A and B be fixed groups, and let $b \to \hat{b}$ be a fixed representation of B as a group of automorphisms of A. Let I be an index set not containing 0 or 1. Then for each $\alpha \in I$, suppose there is a group C_α with two pairings $(\ ,\ )_{\alpha i}$, $i = 1, 2$, of A and B into C_α, each compatible with the representation $b \to \hat{b}$. Then let $G = G(A, B, \{C_\alpha\})$ be the complete direct product of A and B and the $\{C_\alpha\}$, i.e. $G = \{f : \{0, 1\} \cup I \to A \cup B \cup \bigcup_{\alpha\in I} C_\alpha \mid f(0) \in A,\ f(1) \in B,\ f(\alpha) \in C_\alpha$ for all $\alpha \in I\}$.

A product on G may be defined as follows: for $f, g \in G$,

$$(f\cdot g)(\alpha) = \begin{cases} f(0)\ (\widehat{f(1)}\cdot g(0)\ ), & \text{if } \alpha = 0 \\ f(1)g(1), & \text{if } \alpha = 1 \\ (f(0), g(1)\ )_{\alpha 1}(g(0), f(1)\ )_{\alpha 2} f(\alpha)g(\alpha), & \text{for } \alpha \in I. \end{cases}$$

It is then straight forward to verify that G is a group. In the coordinates 0 and 1, G behaves as the ordinary semi-direct product of A and B, and any subgroup involving at most one of the coordinates 0 and 1 is an ordinary direct product in the obvious manner.

The following identities are easily verified.

$$f^{-1}(\alpha)=\begin{cases}\widehat{f(1)}^{-1}f(0)^{-1} & , \alpha=0 \\ f(1)^{-1} \quad , & \alpha=0 \\ (f(0),\ f(1)\ )_{\alpha 1}(f(0),\ f(1)\ )_{\alpha 2}f(\alpha)^{-1}, & \alpha \in I\end{cases}$$

$$(\ [f,g]\ )\ (\alpha)=\begin{cases}f(0)\ (\widehat{f(1)}\cdot g(0)\ )\ (\widehat{f(1)}\cdot g(1)\ )\ (\widehat{f(1)}^{-1}\cdot f(0)^{-1})\ \widetilde{[f(1),g(1)\ ]}\cdot g(0)^{-1}, & \alpha=0 \\ [f(1),g(1)\ ], \quad \alpha=1 & \\ (f(0),g(1)\ )_{\alpha 1}(f(0),g(1)^{-1})_{\alpha 2}(g(0),f(1)^{-1})_{\alpha 1}(g(0),f(1)\ )_{\alpha 2}[g(\alpha),g(\alpha)\ ], & \alpha \in I\end{cases}$$

Here $[g_1,\ g_2]$ denotes the commutator $g_1g_2g_1^{-1}g_2^{-1}$, as usual.

The following example is interesting in that both the representation $b \to \hat{b}$ and the pairing (,) are nontrivial. Let H be a nonabelian group of order p^3, p a prime. Let A be any subgroup of order p^2, and B any subgroup of order p not contained in A. Let C be the commutator subgroup of H and recall that C must also be the center of H and $C \leq A$. For $a \in A$, $b \in B$, put $\hat{b}\cdot a=bab^{-1} \in A$, and $(a,\ b)_1=[a,\ b]=aba^{-1}b^{-1}$. Then

$$(\hat{b}\cdot a,\ \ b_1)_1=(bab^{-1},\ \ b_1)_1=bab^{-1}b_1ba^{-1}b^{-1}b_1^{-1}=(bab^{-1}a^{-1})ab_1a^{-1}(aba^{-1}b^{-1})b_1^{-1}=(a,\ b_1)_1.$$

It is similarly easy to check that $(a,\ b)_1=[a,\ b]$ is a pairing, so it is compatible with the representation $b \to \hat{b}$. Define $(\ ,\ )_2$ by $(a,\ b)_2=e$, the identity element of C. The resulting group $G=G(A,\ B,\ C)$ has order p^4, is nonabelian, and has the following interesting factorization: identify A with $\{\ (a,\ e,\ e)\ \mid\ a \in A$, B with $\{\ (e,\ b,\ e)\ \mid\ b \in B\ \}$, and C with $\{\ (e,\ e,\ c)\ \mid\ c \in C\ \}$. Putting $K=B\cdot C \leq G$ yields $G=A\cdot K=K\cdot A$ where $A \cap K=\{e\}$ and *neither A nor K is normal in G.*

This example can be generalized somewhat, and one rather natural example follows. Let B be an arbitrary subgroup of a group H. Define A_0 by $A_0=\{a \in H\ \mid\ [a,\ b] \in Z(H)$ for all $b \in B\}$. It is easy to check that A_0 is closed under multiplication, so it is a group, at least if it is finite. Let A be any subgroup of A_0 which is invariant under the inner automorphisms of H determined by elements of B. Then $b \to \hat{b}$ defined by $\hat{b}\cdot a=bab^{-1}$ is a representation of B as a group of automorphisms of A. Furthermore, the mapping $(\ ,\ )_1 : A \times B \to Z(C)$ defined by $(a,\ b)_1=[a,\ b]$ is a pairing of A and B into C compatible with the representation $b \to \hat{b}$. The compatibility argument looks just like the one in the above paragraph. (The reader might find helpful in these verifications a commutator formula such as 10.2.1.2 of Hall (1959)).

COSET GEOMETRIES

The construction of projective planes and generalized polygons in general has been of interest to many authors in recent years. Higman and McLaughlin (1961) introduce coset geometries to study properties of planes, and Cronheim (1965) exploits coset geometries (albeit slightly disguised) at length in connection with the study of planes.

Let A and B be subgroups of a group G. Then the coset geometry $\pi=\pi(G,\ A,\ B)$ is defined as follows: left cosets of A (of B) are called lines (points) of π. A line Ax is to be *incident* with the point By if and only if $Ax \cap By \neq 0$. If G is finite, then each line of π lies on $[A : A \cap B]$ points; each point lies on $[B : A \cap B]$ lines. The number of points is $[G : B]$; the number of lines is $[G : A]$.

Higman and McLaughlin (1961) prove that π is a partial plane if and only if $AB \cap BA=A+B$. The same type of argument shows that if both $AB \cap BA=A+B$

and $ABA \cap BAB = AB + BA$, then there are no triangles in π, so π is a partial 4-gon as defined below.

Let F be a finite field with q elements. Put $A = B = C = (F, +)$. Let s, t, u, v be additive homomorphisms of F, i.e. linear transformations of F over its prime subfield. Define $(,)_i : A \times B \to C$, $i = 1, 2$, by $(a, b)_1 = a^s b^t$, $(a, b)_2 = a^u b^v$. Then $(,)_i$ are pairings and the multiplication of $G = G(A, B, C)$ (where $\wedge$ is trivial) is represented explicitly as $(a, b, c) \cdot (x, y, z) = (a+x,\ b+y,\ a^s y^t + b^u x^v + c + z)$. Then $G = G(A, B, C)$ together with the subgroups A, B, and C of G form a T-group, as defined on page 3 of Cronheim (1965). To see the connection between Cronheim's representation and the present one, let the symbol [b, c] denote the line (coset of A) containing the element (0, b, c), and let the symbol (a, c) denote the point containing the element (a, 0, c). Identify A and B with the subgroups of elements of the form (a, 0, 0) and 0, b, 0), respectively. Then it is an easy matter to write out the incidence relation in terms of these elements and to check that π is a partial plane if and only if $a^s b^t + a^v b^u = 0$ implies $ab = 0$. In that case π is clearly an elliptic semi-plane of type (a) in the terminology of Dembowski (1968). In view of the rather lengthy considerations for planes in Albert (1960), as well as in the references already cited, attention will now be centered on partial 4-gons.

A partial 4-gon P is a point-line configuration in which no two points are incident with two lines in common and in which no triangles exist. A generalized 4-gon P_4 is a partial 4-gon such that if x is any point of P_4 not incident with some line L, then there is a (necessarily unique) line L' incident with x and concurrent with L. The following general scheme is hereby proposed for obtaining generalized 4-gons. Let $A = B = C_1 = C_2$ be the additivegroup of the finite field $F = GF(q)$. Define pairings $(,)_{i,j}$, $1 \leq i, j \leq 2$ in terms of linear transformations L_{ij}, R_{ij} of F over its prime subfield as follows:

$$(x, y)_{ij} = x^{L_{ij}} y^{R_{ij}}, \quad \text{for } x, y \in F,\ 1 \leq i,\ j \leq 2.$$

And let $\wedge$ be trivial. Identify the groups A, B, C_1, and C_2 with elements of $G = F^4$ of the forms (a, 0, 0, 0), (0, b, 0, 0), (0, 0, c, 0), and (0, 0, 0, d), respectively. Then multiplication in $G = G(A, B, C_1, C_2)$ appears as

$$(a_1, b_1, c_1, d_1) \cdot (a_2, b_2, c_2, d_2) =$$

$$(a_1+a_2,\ b_1+b_2,\ a_1^{L_{11}} b_2^{R_{11}} + a_2^{L_{12}} b_1^{R_{12}} + c_1 + c_2,\ a_1^{L_{21}} b_2^{R_{21}} + a_2^{L_{22}} b_1^{R_{22}} + d_1 + d_2).$$

The conditions given at the beginning of this section which are necessary and sufficient for $\pi(G, A, B)$ to be a partial 4-gon translate to conditions on the linear transformations as follows. For all $a, b \in F$, if

$$a_1^{L_{i1}} b_1^{R_{i1}} + a_2^{L_{i2}} b_2^{R_{i2}} = a_1^{L_{i2}} b_1^{R_{i2}} + a_2^{L_{i1}} b_2^{R_{i1}}$$

for both $i = 1$ and $i = 2$, then $a_1 a_2 (b_1 + b_2) = 0$.

Suppose eight linear transformations satisfying this condition can be found. In $\pi(G, A, B)$, pick coset representatives of A and B to obtain q^3 points (x, y, z) and q^3 lines [u, v, w] of a partial 4-gon. By adjoining points (b, c), (b), (∞) and lines [b, c], [c], [∞], $b, c \in F$, and then defining incidence suitably, one would hope to obtain a generalized 4-gon.

As an example, suppose

$$L_{11} = L_{21} = R_{11} = \mathrm{id},\ L_{12} = R_{12} = L_{22} = R_{22} = 0,$$

and $R_{21} = \alpha$, for some linear map α. Then the condition given above is satisfied if and only if $F = GF(2^e)$ for some e, and the map $a \to a^\alpha a^{-1}$ is a permutation of the non-zero elements of F. In some cases the resulting partial 4-gons can be completed to generalized quadrangles. Indeed, the quadrangles with an affine representation as described in Payne (1970) include as a subset many of the quadrangles described here. As examples of suitable linear α, there are the automorphisms of F of maximal order e. The resulting partial quadrangles can be completed. As an additional example, suppose $F = F_4 = \{0, 1, \zeta, 1+\zeta\}$, where $\zeta^2 = 1+\zeta$. Let α be the projection defined by $(a+b\zeta)^\alpha = a$, for $a, b \in \{0, 1\}$. Then the map $a \rightarrow a^\alpha a^{-1}$ is a permutation of the non-zero elements of F_4. It is not clear whether or not the corresponding partial quadrangle can be completed to a generalized quadrangle.

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REFERENCES CITED

Albert, A. A. 1960. Finite division algebras and finite planes. Proc. Symposium Appl. Math. 10: 53–70.
Cronheim, A. 1965. T-Groups and their geometry. Ill. J. Math. 9: 1–30.
Dembowski, P. 1968. Finite Geometries. Springer-Verlag, New York. 375 p.
Feit, W., and **G. Higman.** 1964. The nonexistence of certain generalized polygons. J. Algebra 50: 114–131.
Hall, M., Jr. 1959. The Theory of Groups. Macmillan, New York. 434 p.
Higman, D. G., and **J. E. McLaughlin.** 1961. Geometric ABA-groups. Ill. J. Math. 5: 382–397.
MacLane, S. 1963. Homology. Springer-Verlag, Berlin. 422 p.
Payne, S. E. 1970. Affine representations of generalized quadrangles. J. Algebra 16: 473–485.
Sah, Chih, Han. 1968. Automorphisms of finite groups. J. Algebra 10: 47–68.